GORENSTEINNESS OF EHRHART RINGS OF MATCHING POLYTOPES

KOJI MATSUSHITA

This talk is based on a joint work with B. Eisley and A. R. Vindas-Meléndez ([2]). Let $\Bbbk$ be a field and $P \subset \mathbb{R}^d$ a lattice polytope. We define the *Ehrhart ring* of P as follows:

$$A(P) := \Bbbk[t_1^{a_1} \cdots t_d^{a_d} t_{d+1}^n : n \in \mathbb{Z}_{>0}, (a_1, \dots, a_d) \in nP \cap \mathbb{Z}^d].$$

We regard $A(P)$ as a graded ring by setting $\deg(t_1^{a_1} \cdots t_d^{a_d} t_{d+1}^n) = n$. It is known that the Ehrhart ring $A(P)$ is a Cohen–Macaulay semi-standard graded ring with $\dim A(P) = \dim P + 1$. Moreover, $A(P)$ is standard graded if and only if P has the *integer decomposition property* (*IDP*, for short), that is, for any $n \in \mathbb{Z}_{>0}$ and $\alpha \in nP \cap \mathbb{Z}^d$, there exist $\alpha_1, \dots, \alpha_n \in P \cap \mathbb{Z}^d$ with $\alpha = \alpha_1 + \cdots + \alpha_n$.

The *(polar) dual* of P is:

$$P^* := \left\{ y \in \mathbb{R}^d : \langle x, y \rangle \geq -1 \text{ for all } x \in P \right\}.$$

We call P *reflexive* if both P and P^* are lattice polytopes.

Theorem 0.1 ([3], see also [1, Theorem 1.1])**.** *The following are equivalent:*

(i) *The Ehrhart ring $A(P)$ is Gorenstein.*
(ii) *There exist $k \in \mathbb{Z}_{>0}$ and $\alpha \in \mathbb{Z}^d$ such that $kP - \alpha$ is reflexive.*
(iii) *There exist $k \in \mathbb{Z}_{>0}$ and $\alpha \in kP \cap \mathbb{Z}^d$ such that α has height 1 over each facet of P.*

We say that P is *Gorenstein of index k* if $kP + \alpha$ is a reflexive polytope for some $\alpha \in \mathbb{Z}^d$.

In this talk, we characterize when matching polytopes are Gorenstein.

Let G be a simple graph on the vertex set $V(G)$ with the edge set $E(G)$. Let $m \subseteq E(G)$ such that every vertex of G is incident to at most one edge in m. Then we call m a *matching* on G, and identify it with the indicator vector $m \in \mathbb{R}^E$ given by:

$$m(e) := \begin{cases} 1 & \text{if } e \in m, \\ 0 & \text{if else.} \end{cases}$$

The *matching polytope* associated to G is the convex hull of all matchings on G:

$$P_M(G) := \operatorname{conv} \left\{ m \in \mathbb{R}^E : m \text{ is a matching on } G \right\}.$$

Given a vertex $u \in V(G)$, let $\deg_G(u)$ denote the degree of v in G. We say that $u \in V$ is *essential* if one of the following three conditions is satisfied:

(1) $\deg_G(u) = 1$ and the degree of its neighbor is also 1,
(2) $\deg_G(u) = 2$ and its neighbors are not adjacent, or
(3) $\deg_G(u) \geq 3$.

Some graphs that are of relevance to this work include:

- the cycle on n vertices C_n.
- The *chortling cycle* C_5' on 5 vertices is a cycle with two chords (see Figure 1):
$$V(C_5') = \{0, 1, 2, 3, 4\},$$
$$E(C_5') = \{\{i, i+1\} : i \in \{0, 1, 2, 3\}\} \sqcup \{\{0, 4\}, \{1, 3\}, \{2, 4\}\}.$$
- The *complete multipartite graph* $K_{r_1,\dots,r_n}$ $(n > 1)$ is the graph on the vertex set of the form $\bigsqcup_{i=1}^{n} U_i$ ($|U_i| = r_i$ for $i = 1, \dots, n$) with the edge set:
$$E\left(K_{r_1,\dots,r_n}\right) := \left\{\{u, u'\} \colon u \in U_i, u' \in U_j, 1 \le i < j \le n\right\}.$$
When $r_1 = \cdots = r_n = 1$, we denote $K_{\underbrace{1, \dots, 1}_{n}}$ by K_n, and call this the *complete graph* on n vertices.

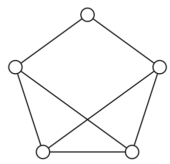

FIGURE 1. The chortling cycle C_5'.

The following is our main theorem:

Theorem 0.2. *Let G be a connected graph. $P_M(G)$ is Gorenstein if and only if G is one of the following graphs:*

(a) *A bipartite graph whose essential vertices have the same degree;*
(b) *C_5;*
(c) *A graph whose essential vertices have the same degree 3 and whose maximal 2-connected components are bipartite, K_3, $K_{1,1,2}$ or C_5';*
(d) *K_4.*

Moreover, if $P_M(G)$ is Gorenstein, then $P_M(G)$ has the IDP.

REFERENCES

[1] Emanuela De Negri and Takayuki Hibi. Gorenstein algebras of Veronese type. *J. Algebra*, 193(2):629–639, 1997.

[2] Benjamin Eisley, Koji Matsushita, and Andrés R. Vindas-Meléndez. Matching polytopes, Gorensteinness, and the integer decomposition property. *Graphs Combin.*, 41(3):Paper No. 58, 14, 2025.

[3] Takayuki Hibi. Dual polytopes of rational convex polytopes. *Combinatorica*, 12(2):237–240, 1992.

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, KOMABA, MEGURO-KU, TOKYO 153-8914, JAPAN

Email address: `koji-matsushita@g.ecc.u-tokyo.ac.jp`